

Final Exam — Partial Differential Equations (WBMA008-05)

Monday 15 June 2026, 15.00h–17.00h

University of Groningen

Instructions

1. The use of calculators is *not* allowed. It is allowed to use a “cheat sheet” with notes (one sheet A4, both sides, handwritten, “wet ink”).
 2. All answers need to be accompanied with an explanation or a calculation: only answering “yes”, “no”, or “42” is not sufficient.
 3. If p is the number of marks then the exam grade is $G = 1 + p/10$.
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Problem 1 (8 + 12 = 20 points)

Solve the following nonuniform transport equation:

$$\frac{\partial u}{\partial t} + x \frac{\partial u}{\partial x} = 0, \quad u(0, x) = f(x).$$

~~(X)~~ Compute *all* characteristic curves.

~~(b)~~ Assume that f is continuous. Show that $\lim_{t \rightarrow \infty} u(t, x) = f(0)$ for all $x \in \mathbb{R}$.

Problem 2 (8 + 8 + 4 = 20 points)

Consider the Laplace equation in polar coordinates:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0, \quad r > 0, \quad \theta \in \mathbb{R}.$$

~~(X)~~ Find all solutions of the form $u(r, \theta) = r^k w(\theta)$, where $k \geq 0$ is an integer.

~~(X)~~ Find a series solution of the above equation which is defined for all $r \geq 0$ and which satisfies the boundary condition $u(\rho, \theta) = h(\theta)$, where $\rho > 0$ and h is a 2π -periodic function. Give expressions for the coefficients that appear in the series.

~~((c))~~ Compute the solution $u(r, \theta)$ when $\rho = 2$ and $h(\theta) = 1 + 8 \cos^2(\theta)$.

~~Problem~~ 3 (20 points)

Recall the following function:

$$G_0(x, y; \xi, \eta) = -\frac{1}{2\pi} \log \|(x, y) - (\xi, \eta)\|,$$

where $\|\cdot\|$ denotes the Euclidean norm. Use the method of images to construct a Green's function for Poisson's equation on the domain $\Omega = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}$.

Hint: use three image points of $(\xi, \eta) \in \Omega$.

Turn over for problems 4 and 5!

Problem 4 (20 points)

Consider the following heat equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad t > 0, \quad -\infty < x < \infty,$$

with initial condition $u(0, x) = f(x)$.

Use Fourier transforms to explicitly determine the function G , depending on both t and x , such that the solution is given by $u(t, x) = (G * f)(x)$.

Hint: recall that $f(x) = (\delta * f)(x)$.

Problem 5 (10 points)

Consider the following equation (which can be seen as either a nonlinear transport equation or Burgers' equation without a diffusion term):

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0.$$

Show that travelling wave solutions of the form $u(t, x) = v(x - ct)$, where v is differentiable, exist if and only if v is constant.

End of test (90 points)